

Full counting statistics for noninteracting fermions: joint probability distributions

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Abstract

The joint probability distribution in the full counting statistics (FCS) for noninteracting electrons is discussed for an arbitrary number of initially separate subsystems which are connected at $t = 0$ and separated again at a later time. A simple method to obtain the leading-order long-time contribution to the logarithm of the characteristic function is presented which simplifies earlier approaches. New explicit results for the determinant involving the scattering matrices are found. The joint probability distribution for the charges in *two* leads is discussed for Y junctions and dots connected to four leads.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The theory of noise in quantum transport in mesoscopic systems is a very active field of research [1, 2]. In addition to the first few moments of the transmitted charge the full probability distribution can be studied, called *full counting statistics* (FCS). This was first done in a publication by Levitov and Lesovik [3] where noninteracting fermions were treated using the cumulant generating function. The system usually studied consists of a finite ‘dot’ region connected to M leads which initially are separated from the dot region and have different chemical potentials [3–6]. After connecting the subsystems the time evolution of the particle transfer between the leads is studied. In order to avoid mathematical subtleties it is useful to start with a *finite* total number of particles N_{tot} which can be achieved using leads of finite extent. The thermodynamic limit is performed only at a later stage. For the lattice models with a finite number of states at each lattice site studied in this paper this implies also a *finite* number N_{H} of the dimension of the Hilbert space of a single fermion.

For an initial state which is a Slater determinant with N_{tot} fermions the characteristic function $g(t)$ for noninteracting fermions is an $N_{\text{tot}} \times N_{\text{tot}}$ determinant. After averaging over a grand canonical ensemble an expression for $g(t)$ in terms of an $N_{\text{H}} \times N_{\text{H}}$ determinant can be derived [4, 7, 8]. In some publications this result is called the ‘Levitov–Lesovik formula’ [9]. This expression is the formal starting point for the actual calculation of the characteristic function. It consists of two steps. The first one is to calculate the time dependence

of the one-particle projection operators $P_a(t)$ onto lead a . For finite times exact results can be obtained numerically [8]. In the long-time limit an accurate analytical approximation can be given in terms of scattering states after performing the thermodynamic limit. In the second step the determinant over the one-particle Hilbert space has to be calculated. After the thermodynamic limit has been performed this is an infinite-dimensional determinant and mathematical care is necessary [10]. Various approaches have been proposed for the evaluation of the determinant. Muzykantskii and Adamov [11] used methods from the theory of singular integral equations to proceed for $M = 2$. In the long-time limit they obtained the leading term for $\ln g(t)$ (linear in t) by the exact solution of a matrix Riemann–Hilbert problem. Their approach provided the first explicit derivation of the result presented by Levitov and Lesovik [3]. Alternatively one can use a formal power series expansion of the logarithm of the determinant [8]. The term linear in time can then easily be identified and the infinite-dimensional determinant can be reduced to an energy integral over the logarithm of an $M \times M$ determinant. A third approach used Szegő’s theorem from the theory of Toeplitz matrices to obtain the term linear in time [12].

For two leads ($M = 2$) the contribution to $\ln g(t)$ linear in t vanishes in the case of perfect transmission and subleading terms logarithmic in time have to be considered. This was studied analytically by Muzykantskii and Adamov [11] by an approximate solution of a more complicated Riemann–Hilbert problem as well as by extensions of Szegő’s theorem [12]. Numerical as well as analytical results in agreement with these findings were presented by one of us [8].

In this paper we generalize and simplify the derivation using the formal power series expansion for $\ln g(t)$. After obtaining the general linear in t contribution the $M \times M$ determinant for the *joint* probability distribution is examined in detail. The $M \times M$ matrix, of which one has to evaluate the determinant, is written as the sum of the unit matrix and a second matrix. This allows us to read off the general expression for the characteristic function for the joint probability distribution of two or more observed charged transfers for arbitrary values of M .

Applications to Y junctions ($M = 3$) and dots with $M = 4$ leads are discussed.

2. General formulation

In the following we consider a system which consists of a finite ‘dot’ region described by the Hamiltonian H_0^{dot} and M leads with the Hamiltonians $H_{0,a}$ with $a = 1, \dots, M$. The leads are initially separated from the dot region. The number of electrons in the initial state are N_0^{dot} and $N_{0,a}$. We assume the initial state $|\Phi(0)\rangle$ to be an eigenstate of H_0^{dot} and $H_{0,a}$:

$$|\Phi(0)\rangle = |E_i^{N_0^{\text{dot}}}\rangle \otimes |E_n^{N_{0,1}}\rangle \otimes \dots \otimes |E_p^{N_{0,M}}\rangle. \quad (1)$$

The time evolution for times greater than zero is described by the Hamiltonian

$$H = H_0^{\text{dot}} + \sum_a H_{0,a} + \sum_a V_a \equiv H_0 + V. \quad (2)$$

The term V which couples the leads with the dot region will be specified later. The probability distribution that Q_1 electrons are transferred to lead 1, Q_2 electrons are transferred to lead 2, etc, after time t when the subsystems are separated again, is given by

$$\begin{aligned} w(t, \{Q\}) &= \langle \Phi(t) | \prod_{a=1}^M \delta[Q_a - (\mathcal{N}_a - N_{0,a})] | \Phi(t) \rangle \\ &= \frac{1}{(2\pi)^M} \int d\lambda_1 \dots d\lambda_M e^{-i \sum_a \lambda_a Q_a} g(t, \{\lambda\}). \end{aligned} \quad (3)$$

Here \mathcal{N}_a is the particle number operator of the lead a and $g(t, \{\lambda\})$ is the characteristic function. With the particle number operators $\mathcal{N}_a(t)$ in the Heisenberg picture g is given by

$$g(t, \{\lambda\}) = \langle \Phi(0) | e^{i \sum_a \lambda_a \mathcal{N}_a(t)} e^{-i \sum_a \lambda_a \mathcal{N}_a} | \Phi(0) \rangle. \quad (4)$$

The fact that the initial state is assumed to be an eigenstate of the particle number operators was used. This expression can be easily generalized to initial statistical operators of the type $\rho_0 = \rho_0^{\text{dot}} \otimes \rho_0^{(1)} \otimes \dots \otimes \rho_0^{(M)}$. An important example are initial grand canonical subensembles with different temperatures and chemical potentials:

$$\rho_0^{(a)} = \frac{e^{-\beta_a (H_{0,a} - \mu_a \mathcal{N}_a)}}{\text{Tr} e^{-\beta_a (H_{0,a} - \mu_a \mathcal{N}_a)}} \quad (5)$$

and ρ_0^{dot} of the same type. Then ρ_0 has the generalized canonical form $\rho_0 = e^{-\tilde{H}_0} / \bar{Z}_0$. Averaging yields for the characteristic function

$$g(t, \{\lambda\}) = \langle e^{i \sum_a \lambda_a \mathcal{N}_a(t)} e^{-i \sum_a \lambda_a \mathcal{N}_a} \rangle, \quad (6)$$

where $\langle \dots \rangle$ denotes the averaging with the statistical operator ρ_0 . This result is also valid for interacting fermions.

For noninteracting fermions the expectation value can be simplified using Klich’s trace formula [7, 8]:

$$\text{Tr}(e^A e^B) = \det(1 + e^A e^B), \quad (7)$$

where A and B are arbitrary one-particle operators in Fock space, and a and b are the corresponding operators in the Hilbert space of a single particle. Therefore the characteristic function can be expressed as a determinant in the one-particle Hilbert space:

$$\begin{aligned} g(t, \{\lambda\}) &= \det[1 + (e^{i \sum_a \lambda_a P_a(t)} e^{-i \sum_a \lambda_a P_a} - 1) \bar{n}_0] \\ &\equiv \det[1 + b(t, \{\lambda\})], \end{aligned} \quad (8)$$

where P_a is the projection operator in the Hilbert space of one particle on the states of the a th lead and $\bar{n}_0 = (e^{\tilde{h}_0} + 1)^{-1}$ is the Fermi operator. It is determined by the Fermi functions describing the initial state. In order to obtain joint probability distributions for arbitrary times the first step is the (numerical) calculation of the Heisenberg operator $P_a(t)$.

In order to study the *long-time* behaviour it is useful to introduce the current operators $j_a \equiv [P_a, h]/i$, where h is the Hamiltonian in the Hilbert space of a single particle, and write $P_a(t)$ as [8]

$$P_a(t) = P_a(0) + \int_0^t j_a(t') dt' \equiv P_a + \delta P_a(t). \quad (9)$$

The operator b in equation (8) can then be expressed as

$$b(t, \{\lambda\}) = \sum_a (e^{i \lambda_a} - 1) \delta P_a(t) e^{-i \sum_{a'} \lambda_{a'} P_{a'} \bar{n}_0}. \quad (10)$$

In order to avoid reflections from the ends of the leads (far away from the dot) in the long-time limit, the thermodynamic limit has to be taken first. In this limit the discrete energy spectrum of the initially occupied standing wave states $|\epsilon_j, a\rangle$ becomes continuous and the trace in the one-particle Hilbert space involves an energy *integration* for the lead states:

$$\text{tr} b = \sum_i \langle \epsilon_i^{\text{dot}} | b | \epsilon_i^{\text{dot}} \rangle + \sum_a \int d\epsilon \langle \epsilon, a | b | \epsilon, a \rangle, \quad (11)$$

with the normalization $\langle \epsilon, a | \epsilon', a' \rangle = \delta_{aa'} \delta(\epsilon - \epsilon')$. Expressed with these states the projection operator onto lead a is

$$P_a = \int d\epsilon |\epsilon, a\rangle \langle \epsilon, a|. \quad (12)$$

Using $\det b = \exp(\text{tr} \ln b)$ in equation (8) the logarithm of the characteristic function is given by

$$\ln g(t, \{\lambda\}) = \text{tr} \ln [1 + b(t, \{\lambda\})]. \quad (13)$$

The thermodynamic limit of the trace operation is defined in equation (11). Mathematical subtleties concerning the existence of the determinant in equation (8) in the thermodynamic limit were discussed recently [10]. In the long-time limit the ‘dot part’ of the trace (see equation (11)) gives a finite contribution which will be neglected in the

following. Alternatively one can avoid the dot states altogether by including them in (part of) the leads [8]. Using $\langle \epsilon, a | \bar{n}_0 | \epsilon', a' \rangle = \delta(\epsilon - \epsilon') \delta_{aa'} f_a(\epsilon)$, where $f_a(\epsilon)$ are the Fermi functions of the leads and the matrix elements of the operator $b(t, \{\lambda\})$ with the lead states are given by

$$\langle \epsilon_1, a_1 | b(t, \{\lambda\}) | \epsilon_2, a_2 \rangle = \sum_a \int_0^t \langle \epsilon_1, a_1 | e^{iht'} j_a e^{-iht'} | \epsilon_2, a_2 \rangle dt' \times d(\lambda_a) e^{-i\lambda_{a_2}} f_{a_2}(\epsilon_2), \quad (14)$$

where $d(\lambda_a) \equiv e^{i\lambda_a} - 1$. Equations (13) and (14) summarize the two tasks in order to obtain the characteristic function in the thermodynamic limit. In the first step the time evolution of the matrix element in equation (14) has to be determined. In the second step the trace in equation (13) has to be performed.

3. Long-time limit

As the current operators j_a only involve operators localized near the dot region one can approximate the time evolution in the matrix elements in equation (14) in the long-time limit by $e^{-iht'} | \epsilon, a \rangle \approx e^{-i\epsilon t'} | \epsilon, a+ \rangle$, where $| \epsilon, a+ \rangle$ is the scattering state with an outgoing boundary condition for the connected system [8]. Then the time integration in equation (14) can be easily performed:

$$\langle \epsilon_1, a_1 | b(t, \{\lambda\}) | \epsilon_2, a_2 \rangle \approx \frac{e^{i(\epsilon_1 - \epsilon_2)t} - 1}{i(\epsilon_1 - \epsilon_2)} \times \sum_a d(\lambda_a) \langle \epsilon_1, a_1 + | j_a | \epsilon_2, a_2 + \rangle e^{-i\lambda_{a_2}} f_{a_2}(\epsilon_2). \quad (15)$$

The fact that a simple analytical expression for the matrix elements has been derived allows systematic approximations for calculating the trace in equation (13). The first step is a formal power series expansion:

$$\ln g(t, \{\lambda\}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{tr} b^n(t, \{\lambda\}). \quad (16)$$

In the evaluation of the trace of b^n the time-dependent prefactor of the sum in equation (15) plays a central role. The product of the factors $(e^{i\epsilon_{j,l}t} - 1)/\epsilon_{j,l}$ with $\epsilon_{j,l} = \epsilon_j - \epsilon_l$ is used to obtain a product of $n - 1$ ‘energy conserving’ delta functions:

$$\frac{e^{i\epsilon_{1,2}t} - 1}{i\epsilon_{1,2}} \frac{e^{i\epsilon_{2,3}t} - 1}{i\epsilon_{2,3}} \dots \frac{e^{i\epsilon_{n,1}t} - 1}{i\epsilon_{n,1}} = \frac{\sin(\epsilon_{1,2}t/2)}{\epsilon_{1,2}/2} \frac{\sin(\epsilon_{2,3}t/2)}{\epsilon_{2,3}/2} \dots \frac{\sin(\epsilon_{n,1}t/2)}{\epsilon_{n,1}/2} \rightarrow (2\pi)^{n-1} t \delta(\epsilon_1 - \epsilon_2) \delta(\epsilon_2 - \epsilon_3) \dots \delta(\epsilon_{n-1} - \epsilon_n). \quad (17)$$

Therefore only *one* energy integration remains and the trace in the full one-particle Hilbert space can be converted to a trace in the M -dimensional space of the lead indices a denoted by $\text{tr}_{(M)}$:

$$\text{tr} b^n(t, \{\lambda\}) \rightarrow \frac{t}{2\pi} \int d\epsilon \text{tr}_{(M)} c^n(\epsilon, \{\lambda\}), \quad (18)$$

where the $M \times M$ matrix $c(\epsilon, \{\lambda\})$ has the matrix elements

$$c_{a_1 a_2}(\epsilon, \{\lambda\}) = 2\pi \sum_a d(\lambda_a) \langle \epsilon, a_1 + | j_a | \epsilon, a_2 + \rangle e^{-i\lambda_{a_2}} f_{a_2}(\epsilon). \quad (19)$$

In contrast to the current matrix elements off-diagonal in energy the diagonal elements in equation (19) can be simply expressed in terms of the scattering matrix $s_{a_1, a_2}(\epsilon)$. As shown in appendix A

$$2\pi \langle \epsilon, a_1 + | j_a | \epsilon, a_2 + \rangle = s_{a_1 a}^\dagger(\epsilon) s_{a a_2}(\epsilon) - \delta_{a_1 a} \delta_{a_2 a} \quad (20)$$

holds [14]. If we define the $M \times M$ matrices $e(\{\lambda\})$ and $f(\epsilon)$ as $e_{aa'}(\{\lambda\}) \equiv e^{i\lambda_a} \delta_{aa'}$ and $f_{aa'}(\epsilon) \equiv f_a(\epsilon) \delta_{aa'}$ the matrix c takes the form (suppressing the energy and λ dependences)

$$c = (s^\dagger e s e^\dagger - 1) f \equiv \tilde{c} f. \quad (21)$$

Now the relation $\ln \det(1 + c) = \text{tr} \ln(1 + c)$ can be used backwards. With equations (16) and (18) this yields in leading time order

$$\ln g(t, \{\lambda\}) = \frac{t}{2\pi} \int d\epsilon \ln \det(1 + c). \quad (22)$$

Subleading corrections increase only logarithmically with time [11, 8, 12]. The leading-order term was correctly given by Levitov and Lesovik [3] without presenting a derivation.

4. Evaluation of the determinant

In order to obtain explicit results for the leading order in time result for $\ln g(t, \{\lambda\})$ the determinant in the integrand of equation (22) has to be calculated:

$$D(\epsilon, \{\lambda\}) \equiv \det(1 + \tilde{c} f) = \det(1 - f + s^\dagger e s e^\dagger f). \quad (23)$$

In both representations one has to calculate a determinant of a matrix which is the *sum* of a *diagonal* matrix and an arbitrary matrix. The second decomposition in equation (23) is usually taken as the starting point [3, 13]. If one is interested in the probability distribution of the transferred charge in a *single* lead or the joint probability distribution of only a *few* (e.g. two) of the Q_j it turns out to be preferable to use the first decomposition in equation (23). Then we can use

$$\det(1 + c) = 1 + \text{tr} c + \sum_{\{i < j\}} \det^{(2)} c^{(2)} + \sum_{\{i < j < k\}} \det^{(3)} c^{(3)} + \dots + \det c, \quad (24)$$

where, for example, $\det^{(3)} c^{(3)}$ denotes a 3×3 subdeterminant of c with the indices given by the summation variables. Because of $c = \tilde{c} f$ the Fermi functions can be factored out and it is sufficient to consider the subdeterminants of \tilde{c} . The matrix elements of \tilde{c} are given by

$$\tilde{c}_{aa'} = \sum_b s_{ab}^\dagger (e^{i(\lambda_b - \lambda_{a'})} - 1) s_{ba'} \equiv \sum_b s_{ab}^\dagger d_{ba'} s_{ba'}. \quad (25)$$

We begin with the discussion of the transferred charge in a *single* lead.

4.1. Generalized Levitov–Lesovik formula

We choose $a = 1$ as the channel index of the observed charge transfer, i.e. λ_1 is different from zero, but all λ_a with $a > 1$ are put to zero. For $a' > 1$ this implies

$$\tilde{c}_{aa'} = \sum_b s_{ab}^\dagger (e^{i\lambda_b} - 1) s_{ba'} = s_{a1}^\dagger (e^{i\lambda_1} - 1) s_{1a'} \equiv s_{a1}^\dagger d_1 s_{1a'}, \quad (26)$$

while for $a' = 1$ one obtains

$$\tilde{c}_{a1} = \sum_{b>1} s_{ab}^\dagger d_1^* s_{b1} = (\delta_{a1} - s_{a1}^\dagger s_{11}) d_1^*. \quad (27)$$

Apart from the additional term d_1^* in \tilde{c}_{11} the columns of the matrix \tilde{c} are all proportional to s_{a1}^\dagger . Therefore all subdeterminants of order larger than two on the rhs of equation (24) vanish. Using $1 - |s_{11}|^2 = \sum_{a \neq 1} |s_{1a}|^2$ the trace term is

$$\text{tr } c = \sum_{a \neq 1} |s_{1a}|^2 (d_1^* f_1 + d_1 f_a). \quad (28)$$

The only non-vanishing 2×2 matrices are from the pairs $(i, j) = (1, j > 1)$. In the first column the part proportional to s_{a1}^\dagger does not contribute and with $d_1 d_1^* = -(d_1 + d_1^*)$ one obtains

$$\det^{(2)} c^{(1a)} = -(d_1 + d_1^*) |s_{1a}|^2 f_1 f_a. \quad (29)$$

Therefore the determinant appearing in equation (22) is given by

$$\det(1 + c) = 1 + \sum_{a \neq 1} |s_{1a}|^2 [d_1 f_a \bar{f}_1 + d_1^* f_1 \bar{f}_a] \equiv 1 + L_1, \quad (30)$$

where $\bar{f}_a = 1 - f_a$. This is the generalized Levitov–Lesovik formula for the leading time contribution to $\ln g(t, \lambda_1, 0, \dots, 0)$. The derivation presented here simplifies an earlier one [8].

4.2. Joint probability distribution for two leads

In this subsection we present the general expression for arbitrary values of M for the characteristic function necessary to calculate the joint probability distribution for *two* observed charge transfers. We evaluate $\det(1 + c)$ for arbitrary values of λ_1 and λ_2 and put all λ_i for $i > 2$ to zero. For $j > 2$ this implies

$$\tilde{c}_{ij} = \sum_l s_{il}^\dagger (e^{i\lambda_l} - 1) s_{lj} = s_{i1}^\dagger d_1 s_{1j} + s_{i2}^\dagger d_2 s_{2j}. \quad (31)$$

The third and all higher columns of the matrix \tilde{c} are proportional to the two column vectors s_{i1}^\dagger and s_{i2}^\dagger . Only the first two columns have an additional contribution. Using $d_{21} - d_1^* = d_2 + d_2 d_1^*$ and $d_{12} - d_2^* = d_1 + d_1 d_2^*$ they are given by

$$\begin{aligned} \tilde{c}_{i1} &= d_1^* \delta_{i1} - s_{i1}^\dagger d_1^* s_{11} + s_{i2}^\dagger (d_2 + d_2 d_1^*) s_{21}, \\ \tilde{c}_{i2} &= d_2^* \delta_{i2} + s_{i1}^\dagger (d_1 + d_1 d_2^*) s_{12} - s_{i2}^\dagger d_2^* s_{22}. \end{aligned} \quad (32)$$

The structure of the matrix \tilde{c} implies that only determinants up to order *four* on the rhs of equation (24) can be different from zero.

Of the fourth-order determinants only those of the $c^{(12ij)}$ with $j > i > 2$ are non-vanishing. As new objects 2×2 determinants formed by scattering matrix elements appear:

$$S_{ij}^{(2)} \equiv s_{1i} s_{2j} - s_{1j} s_{2i}. \quad (33)$$

For $j > i > 2$ they contain the interference effect in the scattering of *two* fermions from the leads i and j to the leads 1 and 2. Using again $d_i d_i^* = -(d_i + d_i^*)$ one obtains

$$\det^{(4)} c^{(12ij)} = (d_1 + d_1^*) (d_2 + d_2^*) |S_{ij}^{(2)}|^2 f_1 f_2 f_i f_j. \quad (34)$$

The evaluation of all the determinants of order three and smaller is straightforward but tedious. It is therefore presented in appendix B. With the abbreviation

$$B_j \equiv \frac{1}{2} (|S_{1j}^{(2)}|^2 - |S_{2j}^{(2)}|^2 - |s_{2j}|^2 + |s_{1j}|^2) \quad (35)$$

the general result for $D(\epsilon, \lambda_1, \lambda_2, 0, \dots, 0) = \det(1 + c)$ is, using the definition in equation (30)

$$\begin{aligned} D &= 1 + L_1 + L_2 + \left[d_1 d_2^* \bar{f}_1 f_2 \left(|s_{12}|^2 + \sum_{j \neq 1,2} B_j f_j \right) \right. \\ &\quad \left. + (1 \leftrightarrow 2) \right] + \frac{1}{2} \sum_{i,j>2} |S_{ij}^{(2)}|^2 (d_1 \bar{f}_1 f_i + d_1^* f_1 \bar{f}_i) \\ &\quad \times (d_2 \bar{f}_2 f_j + d_2^* f_2 \bar{f}_j). \end{aligned} \quad (36)$$

In the double sum the restriction to $i \neq j$ is included by the fact that the $S_{ii}^{(2)}$ vanish.

The generalization to joint probability distributions of more than two charges is obvious but the expressions become rather lengthy. Therefore this will not be discussed further here. Instead we next present the result for *all* $\lambda_i \neq 0$ for a special form of the scattering matrix.

4.3. Separable t -matrix

The special case of a dot consisting of a single level, called a ‘simple star’, is described by a scattering matrix of the form (see appendix C)

$$s_{ij} = \delta_{ij} + \alpha_i \alpha_j^* u. \quad (37)$$

The unitarity of the scattering matrix implies

$$u + u^* + |u|^2 \sum_i |\alpha_i|^2 = 0. \quad (38)$$

In the complex u plane this is the equation of a circle with radius r_α around $(-r_\alpha, 0)$, where $1/r_\alpha = \sum_i |\alpha_i|^2$. For $i \neq j$ this implies the inequality

$$|s_{ij}|^2 \leq \frac{4|\alpha_i|^2 |\alpha_j|^2}{(\sum_i |\alpha_i|^2)^2} \quad (39)$$

used later.

For the calculation of $D(\epsilon, \{\lambda\})$ it is useful to write c as

$$c = (s^\dagger e s - e) e^\dagger f \equiv (s^\dagger e s - e) \tilde{f}. \quad (40)$$

Using the unitarity relation equation (38) and the definition

$$S_j \equiv \sum_l |\alpha_l|^2 (e^{i\lambda_l} - e^{i\lambda_j}) \quad (41)$$

the matrix elements of b defined by $\alpha b \alpha^* \equiv s^\dagger e s - e$, i.e. factoring out the diagonal matrices α and α^* , are given by

$$b_{ij} = (e^{i\lambda_i} - e^{i\lambda_j})u + |u|^2 S_j. \quad (42)$$

It is easy to see that all determinants of submatrices of dimension three and larger vanish. For the $m \times m$ submatrices i and j take values i_l with $i_1 < i_2 < \dots < i_m$. The determinant is unchanged if one subtracts the first row from all the other ones:

$$b_{ij} - b_{i_1 j} = (e^{i\lambda_i} - e^{i\lambda_{i_1}})u. \quad (43)$$

As the second and all higher rows are proportional to each other the subdeterminants of dimension $m \geq 3$ vanish. The 2×2 determinants are readily calculated. Using the unitarity relation equation (38) the result for D takes the simple form

$$\begin{aligned} D(\epsilon, \{\lambda\}) &= 1 + \sum_{i \neq j} |\alpha_i|^2 |\alpha_j|^2 |u|^2 d_{ij} \bar{f}_i f_j \\ &= 1 + \sum_{i \neq j} |s_{ij}|^2 (e^{i(\lambda_i - \lambda_j)} - 1) \bar{f}_i f_j \\ &= 1 + \sum_i L_i + \sum_{i \neq j} |s_{ij}|^2 d_i d_j^* \bar{f}_i f_j. \end{aligned} \quad (44)$$

For the special case of the separable t-matrix the interference terms discussed in section 4.2 and all higher ones vanish.

5. Shot noise

In this section we mainly elucidate our result equation (36) which determines the joint probability distribution for two leads and present separately results for $M = 3$ and $M \geq 4$ in the zero temperature limit. As we present explicit results we have to specify the Hamiltonian introduced in equation (2). In the one-particle Hilbert space the leads are described as nearest-neighbour hopping chains:

$$h_{0a} = - \sum_{m \geq 1} (|m, a\rangle \langle m+1, a| + \text{h.c.}). \quad (45)$$

Two types of dots are considered. For the ‘simple star’ the dot Hamiltonian and the coupling term are given by

$$h_0^{\text{dot}} = V_0 |0\rangle \langle 0|, \quad v = - \sum_{a=1}^M (\tau_a |1, a\rangle \langle 0| + \text{h.c.}). \quad (46)$$

For the dot consisting of a ring of M_{dot} sites pierced by a magnetic flux we have

$$h_0^{\text{dot}} = \sum_{i=1}^{M_{\text{dot}}} V_i |i\rangle \langle i| + \sum_{i=1}^{M_{\text{dot}}} (\tau_{i,i+1} |i\rangle \langle i+1| + \text{h.c.}), \quad (47)$$

where $M_{\text{dot}} + 1$ corresponds to 1. The coupling to the leads is assumed to be

$$v = - \sum_{a=1}^M (\tau_a |1, a\rangle \langle i_a| + \text{h.c.}). \quad (48)$$

5.1. Y junctions

For $M = 3$ the double sum in equation (36) is missing and B_3 defined in equation (35) takes a simpler form as $|S_{23}^{(2)}|^2 = |s_{31}|^2$ and $|S_{13}^{(2)}|^2 = |s_{32}|^2$ holds [3]. Using the unitarity of the scattering matrix equation (36) simplifies to

$$D = 1 + L_1 + L_2 + [d_1 d_2^* \bar{f}_1 f_2 (|s_{12}|^2 \bar{f}_3 + |s_{21}|^2 f_3) + (1 \leftrightarrow 2)]. \quad (49)$$

For the three-leg simple star $|s_{12}|^2 = |s_{21}|^2$ holds and equation (49) reduces to the $M = 3$ version of equation (44) with $\lambda_3 = 0$. At zero temperature equation (49) reduces to $D = 1 + L_1 + L_2$ when lead 1 and 2 have equal chemical potentials. The correlation between the charge transfers to leads 1 and 2 is still present as $\ln g$ is determined by $\ln D$.

The joint probability distribution shown in figure 1 is for the case $\mu_3 = \mu_2 + \Delta\mu$. If the energy dependence of the scattering matrix elements can be neglected in the energy interval $[\mu_2, \mu_3]$ the energy integration in equation (22) can be carried out and the characteristic function g is given by

$$g(t, \lambda_1, \lambda_2, 0) = [1 + |s_{13}|^2 (e^{i\lambda_1} - 1) + |s_{23}|^2 (e^{i\lambda_2} - 1)]^{N_t}, \quad (50)$$

where $N_t = t \Delta\mu / (2\pi)$. For integer values of N_t the λ integrations in equation (3) can easily be performed and the $p_{nm}(t)$ in

$$w(t, Q_1, Q_2) = \sum_{n,m} p_{nm}(t) \delta(Q_1 - n) \delta(Q_2 - m) \quad (51)$$

are given by

$$p_{nm}(t) = \binom{N}{n+m} \binom{n+m}{m} T_1^n T_2^m (1 - T_1 - T_2)^{N-(n+m)}, \quad (52)$$

where the $T_i = |s_{i3}|^2$ are the transmission probabilities from lead 3 to leads 1 and 2.

For the ‘simple star’ with $V_0 = 0$, $\tau_1 = \tau_3 = 1$ the transmission probabilities in the middle of the band are given by (see appendix C)

$$T_1 = \frac{4}{(2 + |\tau_2|^2)^2}, \quad T_2 = \frac{4|\tau_2|^2}{(2 + |\tau_2|^2)^2}. \quad (53)$$

For $\tau_2 = 0$ the connected system corresponds to an infinite ideal chain with perfect transmission $T_1 = 1$. If the coupling to lead 2 is switched on T_1 decreases and equals $T_2 = 4/9$ for equal coupling $\tau_2 = 1$.

Figure 1 shows the probabilities $p_{nm}(t)$ for $N_t = 20$ and three values of τ_2 . For $\tau_2 = 0$ the open circle shows the ‘vanishing shot noise’ which is an artefact of the leading order in t approximation for $\ln g$. The neglected logarithmic corrections convert the single delta function to a Gaussian with a width proportional to $(\ln N_t)^{1/2}$ for $N_t \gg 1$ [11, 8, 12]. For a weak coupling to lead 2 ($\tau_2 = 0.5$, dark dots) the anticorrelation between the transported charges to leads 1 and 2 is clearly visible. It was discussed previously on the level of moments [1]. The light dots correspond to the case of equal couplings to the star.

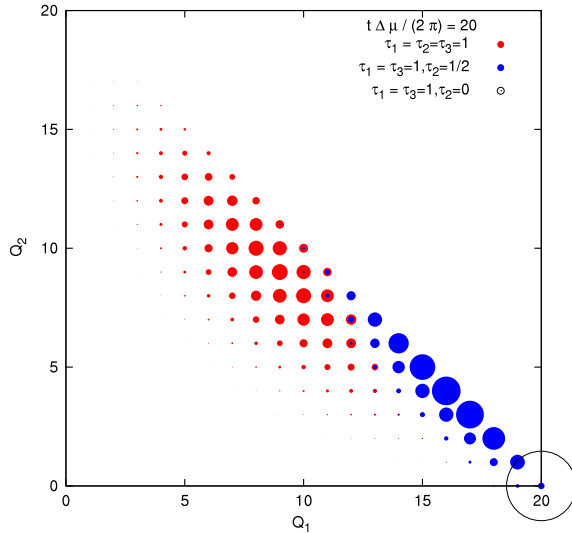


Figure 1. Joint probability distribution for a three-leg simple star for $N_t = 20$ and three different coupling strengths τ_2 to lead 2 using the leading-order time approximation for $\ln g$. The area of the dots is proportional to the p_{nm} in equation (52).

5.2. Systems with $M \geq 4$ leads

For Y junctions there are no interference effects in two-particle scattering processes. The double sum term on the rhs of equation (36) only contributes for $M \geq 4$. To simplify the discussion we here discuss only the special case $\mu_1 = \mu_2 \equiv \mu_R$, i.e. $f_1 = f_2 \equiv f_R$ at zero temperature which implies $f_R^2 = f_R$. Then equation (36) simplifies to

$$D = 1 + L_1 + L_2 + \frac{1}{2} \sum_{i,j>2} |s_{ij}^{(2)}|^2 (d_1 d_2 \bar{f}_R f_i f_j + d_1^* d_2^* f_R \bar{f}_i \bar{f}_j). \quad (54)$$

Already for $M = 4$ there are various possibilities for the chemical potentials of the leads 3 and 4. The simplest one is to assume them to be equal. For arbitrary $M \geq 4$ and $\mu_i = \mu_L = \mu_R + \Delta\mu$ for all $i > 2$ equation (54) further simplifies for the energy interval $\mu_R < \epsilon < \mu_L$ to

$$D = 1 + \sum_{i=1,2} d_i \sum_{j>2} |s_{ij}|^2 + d_1 d_2 \sum_{j>i>2} |s_{ij}^{(2)}|^2. \quad (55)$$

The interference terms in the double sum also occur in the distribution of the total charge $Q_1 + Q_2$ in the leads with chemical potential μ_R which can be obtained by putting $\lambda_1 = \lambda_2 = \lambda$. As the prefactor $d^2 = (e^{i\lambda} - 1)^2$ has no contribution linear in λ the interference terms only enter the cumulants κ_i with $i \geq 2$. When the energy dependence of the scattering matrix for $\mu_R < \epsilon < \mu_L$ is neglected the probability distribution for integer $N_t = t\Delta\mu/(2\pi) \gg 1$ is given by

$$w^{(S)}(t, Q_1 + Q_2) = \sum_{n \geq 0} p_n^{(S)}(t) \delta(Q_1 + Q_2 - n) \quad (56)$$

with

$$p_n^{(S)} = \sum_{l=l_0}^{[n/2]} \binom{N_t}{n-l} \binom{n-l}{l} (1-A+B)^{N_t-n+l} B^l (A-2B)^{n-2l}, \quad (57)$$

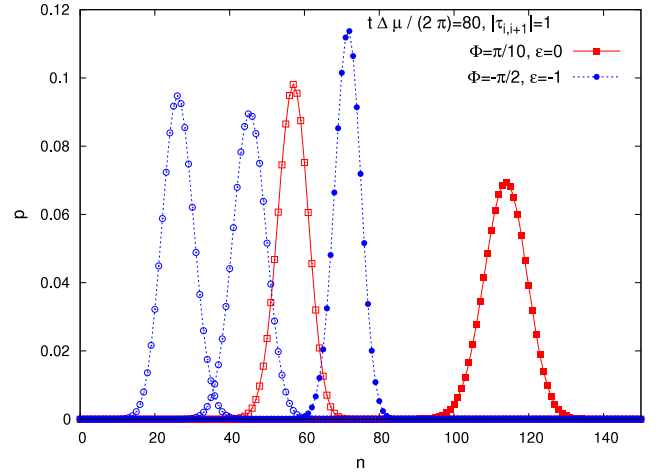


Figure 2. Probability distributions for a four-site ring pierced by a magnetic flux Φ . The full symbols present the $p_n^{(S)}$ of the probability distribution of the sum $Q = Q_1 + Q_2$. The open symbols show the probability distribution for the charge transfer to a single lead. The lines are a guide to the eye. Because of the magnetic flux the distributions for lead 1 and 2 (the two curves with open circles) differ except in the particle-hole symmetric case (open squares).

where $l_0 = \max(0, n - N_t)$ and A and B are given by

$$A = \sum_{i=1}^2 \sum_{j>2} |s_{ij}|^2, \quad B = \sum_{j>i>2} |s_{ij}^{(2)}|^2. \quad (58)$$

In figure 2 we show results for the $p_n^{(S)}$ for a symmetric ring dot with $\tau_a = 0.5$, $V_a = 0$ and $\tau_{i,i+1} = e^{i\Phi/M_d}$, where Φ is the magnetic flux through the ring. We choose $M = M_d = 4$ with each ring site connected to one lead. The scattering matrix elements needed are presented in appendix C. In addition to the probability distribution $w^{(S)}$ of the sum $Q_1 + Q_2$ (filled symbols) we show the probability distributions for the charge transfer to the single leads 1 and 2 (open symbols) which follow from equation (30). The circles correspond to the case when the chemical potentials are close to the centre of the band. Because $|s_{12}|^2 = |s_{14}|^2$ holds for $\epsilon = 0$ for all values of Φ the probability distributions for the charge transfer to leads 1 and 2 are identical. In the generic case they are different for $\Phi \neq 0 \pmod{2\pi}$ as shown for $\epsilon = -1$ (circles). For the parameters chosen it is clearly visible that the sum of the widths of these single charge probability distributions is smaller than the width of $w^{(S)}$. This is another manifestation of the anticorrelation effect [1] mentioned in the discussion of figure 1.

The joint probability distribution $w(t, Q_1, Q_2)$ corresponding to the parameter values used in figure 2 is shown in figure 3. In contrast to figure 1 where the asymmetry in the (dark dot) distribution resulted from asymmetric couplings, the asymmetry of the $\epsilon = -1$ distribution in figure 3 is due to the magnetic flux.

As a simple application of equation (44) for the separable t-matrix we consider the distribution of the total charge $Q_R = Q_1 + \dots + Q_{M_R}$ transferred to the first M_R channels which are all assumed to have the chemical potential μ_R . For the case where the other $M_L = M - M_R$ channels have chemical

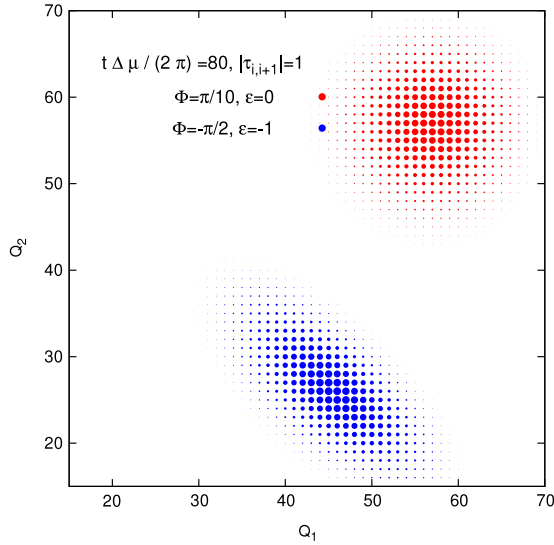


Figure 3. Joint probability distributions for a four-site ring for the parameters used in figure 2.

potential $\mu_L = \mu_R + \Delta\mu$ equation (44) reduces to the $M = 2$ Levitov–Lesovik formula equation (30) with the effective transmission probability

$$T_{\text{eff}} = \sum_{i=1}^{M_R} \sum_{j=M_R+1}^M |s_{ij}|^2 \leq 1. \quad (59)$$

The inequality follows from equation (39) or the explicit result for the simple star in appendix C.

6. Summary

A simple derivation of the leading time order result for the logarithm of the characteristic function which determines the full counting statistics for systems of noninteracting fermions was presented. The energy-dependent determinant involving the scattering matrix of the M -leads system was simplified analytically for three cases. For the distribution of charge transfer to a single lead only the absolute values of the scattering matrix elements enter. For the joint probability distribution of the charges in two leads interference effects in the scattering of pairs of particles become important for $M \geq 4$. For a separable scattering matrix which describes a simple star-like geometry the $M \times M$ determinant was evaluated for joint probability distributions for an arbitrary number of observed charges. All interference terms vanish and only the absolute values of the scattering matrix elements enter as in the generalized Levitov–Lesovik formula.

Explicit results for the probability distributions were presented for the simple star and a ring pierced by a magnetic flux at zero temperature. Various manifestations of the anticorrelation effects in the charge transfer to the observed leads were shown.

Finite temperature effects will be discussed in a forthcoming publication.

Appendix A. Proof of equation (20)

In this appendix we present proof of the relation for current matrix elements with equal energies [14].

The scattering states $|\epsilon, a+\rangle$ introduced in section 3 obey the Lippmann–Schwinger equation [15]

$$|\epsilon, a+\rangle = |\epsilon, a\rangle + g_0(\epsilon + i0)v|\epsilon, a+\rangle, \quad (A.1)$$

where v is the operator which describes the connection of the leads with the dot region and $g_0(z) = (z - h_0)^{-1}$ is the unperturbed resolvent.

As the projection operators P_a commute with h_0 the current matrix elements are given by

$$\langle \epsilon, a_1 + | j_a | \epsilon, a_2 + \rangle = \frac{1}{i} \langle \epsilon, a_1 + | [P_a, v] | \epsilon, a_2 + \rangle. \quad (A.2)$$

Using equation (12) for P_a the first term of the commutator is

$$\langle \epsilon, a_1 + | P_a v | \epsilon, a_2 + \rangle = \int d\epsilon' \langle \epsilon, a_1 + | \epsilon', a \rangle \langle \epsilon', a | v | \epsilon, a_2 + \rangle. \quad (A.3)$$

With the help of the Lippmann–Schwinger equation the overlaps of the unperturbed states with the scattering states can be written as

$$\langle \epsilon, a_1 + | \epsilon', a \rangle = \delta(\epsilon - \epsilon') \delta_{a_1 a} + \frac{\langle \epsilon, a_1 + | v | \epsilon', a \rangle}{\epsilon - \epsilon' - i0}. \quad (A.4)$$

This yields

$$\langle \epsilon, a_1 + | P_a v | \epsilon, a_2 + \rangle = \langle \epsilon, a | v | \epsilon, a_2 + \rangle \delta_{a_1 a} + \int d\epsilon' \frac{\langle \epsilon, a_1 + | v | \epsilon', a \rangle \langle \epsilon', a | v | \epsilon, a_2 + \rangle}{\epsilon - \epsilon' - i0}. \quad (A.5)$$

The matrix element of $v P_a$ can be calculated correspondingly. In the second term in equation (A.5) the $-i0$ in the denominator is replaced by $i0$. This leads to a delta function for the commutator and the energy integration can be carried out. One obtains

$$\langle \epsilon, a_1 + | j_a | \epsilon, a_2 + \rangle = i \langle \epsilon, a_1 + | v | \epsilon, a \rangle \delta_{a_2 a} - i \langle \epsilon, a | v | \epsilon, a_2 + \rangle \delta_{a_1 a} + 2\pi \langle \epsilon, a_1 + | v | \epsilon, a \rangle \langle \epsilon, a | v | \epsilon, a_2 + \rangle. \quad (A.6)$$

With the definition of the scattering matrix $s_{aa'}(\epsilon)$ for potential scattering [15]

$$s_{aa'}(\epsilon) = \delta_{aa'} - 2\pi i \langle \epsilon, a | v | \epsilon, a' + \rangle \equiv \delta_{aa'} - 2\pi i t_{aa'}(\epsilon) \quad (A.7)$$

the validity of equation (20) directly follows.

The unitarity of the scattering matrix can be explicitly confirmed for the multi-lead system [8].

Appendix B. Derivation of equation (36)

In this appendix we discuss the $m \times m$ subdeterminants of the matrix \tilde{c} defined in equation (21) for $\lambda_i = 0$ for all $i \geq 3$. They are characterized by the index set $i_n \in [1, M]$ with $i_1 < i_2 < \dots < i_m$. The corresponding submatrices are denoted by $\tilde{c}^{(i_1, i_2, \dots, i_m)}$. The matrix elements of these submatrices can be

directly read off from the matrix elements of the full $M \times M$ matrix \tilde{c} :

$$\tilde{c}_{ij} = \delta_{i1}\delta_{1j}d_1^* + \delta_{i2}\delta_{2j}d_2^* + s_{i1}^\dagger\alpha_j + s_{i2}^\dagger\beta_j, \quad (\text{B.1})$$

where the α_j and β_j are given in equations (31) and (32). Like the full matrix \tilde{c} all columns of the submatrices are linear combinations of at most the four column vectors δ_{i1} , δ_{i2} , s_{i1}^\dagger and s_{i2}^\dagger . Therefore all subdeterminants with $m > 4$ vanish. In the following we discuss the cases $m = 4, 3, 2$ separately.

$m = 4$: Only for the index set $(1, 2, i, j)$ with $2 < i < j$ do all four different column vectors appear. Therefore all other subdeterminants vanish. For the determinant of $\tilde{c}^{(1,2,i,j)}$ the following formula is used. Let $a \equiv (s_{i1}^\dagger, s_{j1}^\dagger)^T$ and $b \equiv (s_{i2}^\dagger, s_{j2}^\dagger)^T$ be two column vectors. Then the 2×2 determinant of linear combinations of the two vectors is given by

$$\begin{aligned} \det(a\alpha_i + b\beta_i, a\alpha_j + b\beta_j) &= (\alpha_i\beta_j - \alpha_j\beta_i) \det(a, b) \\ &= (\alpha_i\beta_j - \alpha_j\beta_i)(a_i b_j - a_j b_i). \end{aligned} \quad (\text{B.2})$$

The use of this relation and equations (31) and (32) yield equation (34).

$m = 3$: All subdeterminants of \tilde{c}^{ijk} with $2 < i < j < k$ vanish. For $i = 1$ or 2 and $2 < j < k$ the calculation proceeds using equation (B.2):

$$\det \tilde{c}^{(2jk)} = d_1 d_2 d_2^* |s_{j1} s_{k2} - s_{k1} s_{j2}|^2 = -d_1 (d_2 + d_2^*) |S_{jk}^{(2)}|^2. \quad (\text{B.3})$$

In $\tilde{c}^{(12i)}$ with $i > 2$ the first two columns $l = 1, 2$ are of the type $\alpha_l a + \beta_l b + d_l^* e_l$, where e_l is the l th unit column vector while in the third column $\alpha_i a + \beta_i b$ this additional contribution is missing. Therefore the determinants $\det(e_1, a, b)$ and $\det(e_1, e_2, a(b))$ have to be evaluated. This leads to

$$\det \tilde{c}^{(12i)} = (d_1 + d_1^*) d_2^* (|S_{2i}^{(2)}|^2 - |s_{1i}|^2) + (1 \leftrightarrow 2). \quad (\text{B.4})$$

For $m = 2$ arbitrary combinations of $i < j$ in $\det \tilde{c}^{(ij)}$ contribute. The evaluation is straightforward using equation (B.2).

In order to obtain the form presented in equation (36) the following ‘sum rules’ for the $|S_{ij}^{(2)}|^2$ were used which follow from the unitarity of the scattering matrix:

$$\begin{aligned} \sum_{i>2} |S_{ij}^{(2)}|^2 &= |s_{1j}|^2 + |s_{2j}|^2 - |S_{1j}^{(2)}|^2 - |S_{2j}^{(2)}|^2, \\ \sum_{j>i>2} |S_{ij}^{(2)}|^2 &= 1 + |S_{12}^{(2)}|^2 - |s_{11}|^2 - |s_{22}|^2 - |s_{12}|^2 - |s_{21}|^2. \end{aligned}$$

Appendix C. Results for the scattering matrices

In this appendix the scattering matrices for the two dot models presented in section 5 are calculated. The t -matrix defined in equation (A.7) can be expressed via the resolvent operator $g(z) = (z - h)^{-1}$ as [15]

$$t_{aa'}(\epsilon) = \langle \epsilon, a | v | \epsilon, a' \rangle + \langle \epsilon, a | v g(\epsilon + i0) v | \epsilon, a' \rangle. \quad (\text{C.1})$$

For both dot models discussed in section 5 the first term on the rhs vanishes. With $|\langle \epsilon, a | 1, a \rangle|^2 = \sqrt{1 - (\epsilon/2)^2}/\pi$ this yields

$$s_{aa'}(\epsilon) = \delta_{aa'} - 2i\tau_a \tau_{a'} \sqrt{1 - (\epsilon/2)^2} \langle i_a | g(\epsilon + i0) | i_{a'} \rangle. \quad (\text{C.2})$$

For the simple dot all $|i_a\rangle$ are given by $|0\rangle$ and the scattering matrix is of the separable form discussed in section 4.3.

Using the projection onto the states on the ring the full resolvent matrix elements in equation (C.2) can be written as [8]

$$\langle i_a | g(z) | i_{a'} \rangle = \langle i_a | [z - h_0^{\text{dot}} - \gamma(z)]^{-1} | i_{a'} \rangle \quad (\text{C.3})$$

with

$$\gamma(z) = g_b^0(z) \sum_{a=1}^M |\tau_a|^2 |i_a\rangle \langle i_a|, \quad (\text{C.4})$$

where h_0^{dot} is the one-particle Hamiltonian of the ring and $g_b^0(z)$ is the diagonal element of the resolvent of the semi-infinite chain at the boundary. For $z = \epsilon \pm i0$ it is given by

$$g_b^0(\epsilon \pm i0) = (\epsilon \mp i\sqrt{4 - \epsilon^2})/2. \quad (\text{C.5})$$

The matrix inversion in equation (C.3) is trivial for the simple star because $M_{\text{dot}} = 1$:

$$\langle 0 | g(z) | 0 \rangle = \frac{1}{z - V_0 - \sum_a |\tau_a|^2 g_b^0(z)}. \quad (\text{C.6})$$

For a symmetric ring with $M = M_{\text{dot}}$, $\tau_a = \tau$, $V_a = V_0$ and $\tau_{i,i+1} = \tau_{\text{dot}} e^{i\Phi/M}$ the matrix inversion is straightforward using plane wave states on the dot [16]:

$$\begin{aligned} \langle i_m | g(z) | i_n \rangle &= \frac{1}{M} \\ &\times \sum_{j=1}^M \frac{e^{ik_j(m-n)}}{z - V_0 - |\tau|^2 g_b^0(z) + 2\tau_{\text{dot}} \cos(k_j + \Phi/M)}, \end{aligned}$$

where $k_j = 2\pi j/M$.

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